Universal partial orders

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Assume Γ is a class of complete boolean algebras and Θ is a family of complete homomorphisms between the elements of Γ closed under composition and which contains all identity maps.

Definition

$\mathbb{U}^{\Gamma,\Theta}$ denote the category whose objects are complete boolean algebras in Γ and whose arrows are given by complete homomorphisms $i : \mathbb{B} \to \mathbb{C}$ in Θ.

We say that:

- $\mathbb{B} \leq_\Theta \mathbb{C}$ if there is $i : \mathbb{B} \to \mathbb{C}$ in Θ.
- $\mathbb{B} \leq^*_\Theta \mathbb{C}$ if there is an injective $i : \mathbb{B} \to \mathbb{C}$ in Θ.

We shall be interested just in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$ (or to the “equivalent” $\mathbb{U}^{\mathsf{SP},\mathsf{SP}}$) for reason that will be soon transparent.

OPEN PROBLEM Can our methods work for other nice category forcings (proper, CCC, axiom A)?

Remark

If $\mathbb{U}^{\Gamma,\Theta}$ has the amalgamation property then it is a trivial forcing notion since all conditions are compatible.

We like anti-Ramsey classes of partial orders.

Remark

For any cba $\mathbb{B}$ there is a regular embedding $i : \mathbb{B} \to Coll(\omega, \delta)$ for any large enough δ.

Fact

The class of all cbas and all complete homomorphisms between them is a trivial partial order.

Definition

$\mathbb{B}$ is stationary set preserving SSP if

$$[\![S \text{ is stationary }]\!]_{\mathbb{B}} = 1_{\mathbb{B}}$$

for all S stationary subset of ω_1 in V.

Definition

A complete homomorphism (now it is important that i may not be injective) $i : \mathbb{B} \to \mathbb{C}$ is SSP-correct iff

$$[\![\mathbb{C}/i[\dot{G}_{\mathbb{B}}] \in \mathrm{SSP}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

$\mathbb{U}^{\mathrm{SSP},\mathrm{SSP}}$ is the category of complete SSP boolean algebras with SSP-correct homomorphisms.

Fact

$\mathbb{U}^{\text{SSP,SSP}}$ *has no minimal elements.*

Assume

- P is Namba forcing on $\aleph_2$,
- Q is $Coll(\omega_1, \omega_2)$.

Then $\text{RO}(P), \text{RO}(Q)$ are incompatible conditions in $\mathbb{U}^{\text{SSP,SSP}}$.

If $\mathbb{D} \leq \mathrm{RO}(P), \mathrm{RO}(Q)$, and H is V-generic for $\mathbb{D}$:

- $\omega_1^{V[H]} = \omega_1$,
- there are $G, K \in V[H]$ V-generic filters for P and Q respectively (since $\mathbb{D} \leq \mathrm{RO}(P), \mathrm{RO}(Q)$).

G gives in $V[H]$ a sequence cofinal in ω_2^V of type ω.

K give in $V[H]$ a sequence cofinal in ω_2^V of type $(\omega_1)^V$.

Contradiction with the preservation of ω_1 in $V[H]$.
This argument can be repeated in $V^{\mathbb{B}}$ for any $\mathbb{B} \in \mathrm{SSP}$.

Total rigidity and closure

Definition

Given Γ, Θ as required $\mathbb{B} \in \Gamma$ is Θ-totally rigid if for $i_0, i_1 : \mathbb{B} \to \mathbb{C}$ in Θ we have that $i_0 = i_1$.

Definition

$\mathbb{U}^{\Gamma,\Theta}$ is Ord-closed if every set sized descending sequence in $\leq^*_\Theta$ has a lower bound in Γ.

Closure is granted by iteration theorems...

Remark

Total rigidity and closure are the key properties of these class category forcings in order to prove nice properties about their structure.

Universality of $\mathbb{U}^{\Gamma,\Theta}$ and total rigidity

We would like that $\mathbb{U}^{\Gamma,\Theta}$ is universal for Γ.

The optimal case is that there is a complete embedding

$$i_{\mathbb{B}} : \mathbb{B} \to \mathbb{U}^{\Gamma,\Theta} \restriction \mathbb{B}$$

for a dense set of $\mathbb{B} \in \mathbb{U}^{\Gamma,\Theta}$.

If this is the case, take $\mathbb{C} \in \Gamma$, find $\mathbb{B} \leq_{\Theta} \mathbb{C}$ in the above dense set and $i : \mathbb{C} \to \mathbb{B}$ in Θ.

Then $i_{\mathbb{B}} \circ i : \mathbb{C} \to \mathbb{U}^{\Gamma,\Theta} \restriction \mathbb{B}$ will witness that $\mathbb{U}^{\Gamma,\Theta} \restriction \mathbb{B}$ absorbs $\mathbb{C}$ as well.

We have a natural candidate for an $i_{\mathbb{B}} : \mathbb{B} \to \mathbb{U}^{\Gamma,\Theta} \restriction \mathbb{B}$:

$$b \in \mathbb{B} \mapsto \mathbb{B} \restriction b.$$

- $i_{\mathbb{B}}$ is order preserving,
- $i_{\mathbb{B}}$ preserves maximal antichains: if $\{a_i : i \in I\} \subset \mathbb{B}^+$ is a maximal antichain then
$$\prod_{i \in I} (\mathbb{B} \restriction a_i)$$
(the lottery sum of $\{\mathbb{B} \restriction a_i : i \in I\}$) is isomorphic to $\mathbb{B}$. This is the top element of $\mathbb{U}^{\Gamma,\Theta} \restriction \mathbb{B}$.
- PROBLEM: Does this map preserve incompatibility? In general NO!!! Take $\mathbb{B}$ to be homogeneous to get counterexamples........

We need to destroy the homogeneity of $\mathbb{B}$ to hope that $i_{\mathbb{B}}$ is a complete embedding.

Lemma

Let Γ and Θ be as required and $\mathbb{B} \in \Gamma$. TFAE:

- *For all $\mathbb{C} \leq_\Theta \mathbb{B}$ there is just one $i : \mathbb{B} \to \mathbb{C}$ in Θ.*
- *$\mathbb{B} \upharpoonright b$ and $\mathbb{B} \upharpoonright \neg b$ are incompatible in $\mathbb{U}^{\Gamma,\Theta}$ for all $b \in \mathbb{B}^+$.*

This Lemma gives that totally rigid cbas $\mathbb{B} \in \Gamma$ are absorbed by $\mathbb{U}^{\Gamma,\Theta}$ using the map $i_{\mathbb{B}}(b) = \mathbb{B} \upharpoonright b$. Since for these posets the map preserves incompatibility.

Sketch of proof:

Let $i_j : \mathbb{B} \to \mathbb{C}$ ($j = 0, 1$) be distinct arrows in Θ and H be V-generic for $\mathbb{C}$.
Then $G_0 = i_0^{-1}[H] \neq G_1 = i^{-1}[H]$. So there is $b \in G_0 \setminus G_1$.
Notice that $i_0(b), i_1(\neg b) \in H$, thus

$$q = i_0(b) \wedge i_1(\neg b) \in H$$

and thus is positive. Define

- $k_0 : \mathbb{B} \restriction b \to \mathbb{C} \restriction q$
  $k_0(c) = i_0(c) \wedge q$.
- $k_1 : \mathbb{B} \restriction \neg b \to \mathbb{C} \restriction q$
  $k_1(c) = i_1(c) \wedge q$.

Then $k_0, k_1 \in \Theta$ witness that $\mathbb{B} \restriction \neg b$ is compatible with $\mathbb{B} \restriction b$ in $\mathbb{U}^{\Gamma,\Theta}$.
The converse implication is not much harder.

Freezeability

Actually we need less than the density of totally rigid posets to get the universality of $\mathbb{U}^{\Gamma,\Theta}$.

Lemma

Given Γ, Θ as required and $k : \mathbb{B} \to \mathbb{C}$ in Θ, TFAE:

- *The map*
$$i : \mathbb{B} \to \mathbb{U}^{\Gamma,\Theta} \restriction \mathbb{C}$$
given by
$$b \mapsto \mathbb{C} \restriction k(b)$$
is a complete embedding.
- *For all $b \in \mathbb{B}$, $\mathbb{C} \restriction k(b)$ and $\mathbb{C} \restriction k(\neg b)$ are incompatible in $\mathbb{U}^{\Gamma,\Theta}$.*
- *$i_0 \circ k = i_1 \circ k$ for all $i_j : \mathbb{C} \to \mathbb{D}$ ($j = 0, 1$) in Θ.*

$k : \mathbb{B} \to \mathbb{C}$ freezes $\mathbb{B}$ if it satisfies any of the above requirements.

Theorem

Assume $\mathbb{B} \in$ SSP. Then there is an SSP-correct regular embedding $k : \mathbb{B} \to \mathbb{C}$ which freezes $\mathbb{B}$.

Now we can combine the above result on $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$ with the closure properties of $\mathbb{U}^{\mathsf{SP},\mathsf{SP}}$ and the identification of these two categories modulo a dense subset (and large cardinal axioms) and get the following results:

Theorem

Assume there are class many supercompact cardinals. Then the class of totally rigid partial orders is dense in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$.

Theorem

Assume δ is an inaccessible limit of $< \delta$-supercompact cardinals. Then $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}} \cap V_\delta = \mathbb{U}_\delta$ is totally rigid and stationary set preserving.

Moreover the proofs give that $\mathbb{U}_\delta$ is a very nice universal partial order for SSP $\cap V_\delta$ since for each $\mathbb{B}$ in $\mathbb{U}_\delta$ we can find an SSP-correct $i : \mathbb{B} \to \mathbb{U}_\delta \restriction \mathbb{B}$.

Sketch of proof of the density of totally rigid cbas in $\mathbb{U}^{\text{SSP,SSP}}$

Define
$$\mathcal{F} = \{k_{\alpha,\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \omega_1\}$$
by recursion on ω_1.

Given α limit and countable let:

- $\mathbb{B}_{\alpha+2n+1} = \mathbb{B}_{\alpha+2n} * \dot{C}oll(\omega_1, < \kappa)$ where κ is supercompact.
- $k_{\alpha+2n+1,\alpha+2n+2} : \mathbb{B}_{\alpha+2n+1} \to \mathbb{B}_{\alpha+2n+2}$ be a regular embedding which freezes $\mathbb{B}_{\alpha+2n+1}$.
- $\mathbb{B}_\alpha = \text{RCS}(\mathcal{F} \upharpoonright \alpha)$.

Then $C(\mathcal{F}) \leq \mathbb{B}_0$ is SSP and totally rigid.

- $C(\mathcal{F})$ is SSP:

  $Coll(\omega_1, < \kappa)$ forces the equality SSP $=$ SP.
  In the generic extensions $V[G_{\alpha+2n+1}]$ by generics for the odd stages of the iteration we get that $\mathcal{F}/G_{\alpha+2n+1}$ is a semiproper iteration.
  Thus $C(\mathcal{F})/G_{\alpha+2n+1}$ is semiproper and we are done.
- $C(\mathcal{F})$ is totally rigid.

Why $C(\mathcal{F})$ is totally rigid:

Assume $C(\mathcal{F}) \restriction f$ is compatible with $C(\mathcal{F}) \restriction \neg f$ in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$.

Since f is a thread we get that for all α:

$\mathbb{B}_\alpha \restriction f(\alpha) \geq_{\mathsf{SSP}} C(\mathcal{F}) \restriction f$
$\mathbb{B}_\alpha \restriction \neg f(\alpha) \geq_{\mathsf{SSP}} C(\mathcal{F}) \restriction \neg f$.

Thus for all $\alpha < \omega_1$

$\mathbb{B}_\alpha \restriction \neg f(\alpha)$ and $\mathbb{B}_\alpha \restriction \neg f(\alpha)$

are compatible conditions in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$.

This is impossible since $k_{\alpha,\alpha+2}$ freezes $\mathbb{B}_\alpha$ for all α.

Thus $k_{\alpha,\alpha+2} \circ f(\alpha) = f(\alpha + 2)$ gives that

$\mathbb{B}_{\alpha+2} \restriction f(\alpha + 2)$ is NOT compatible with $\mathbb{B}_{\alpha+2} \restriction \neg f(\alpha + 2)$.

What is $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$ useful for?

Theorem

Assume

$T \supseteq \mathsf{ZFC}+\mathsf{MM}^{+++}+\{p \subset \omega_1\}+\{$ *there are class many superhuge cardinals*$\}$.

Then for any formula $\phi(x)$ *TFAE:*

1. $T \vdash (H_{\omega_2} \models \phi(p))$
2. T *proves that there is* $\mathbb{B} \in \mathsf{SSP}$ *such that*

$$[\![\mathsf{MM}^{+++} \wedge \phi(p)^{H_{\omega_2}}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

Forcing becomes a proof booster for theorems over extensions of ZFC by strong forcing axioms.
Actually the theorem gives a completeness theorem for the theory of H_{ω_2} with respect to the semantic given by boolean valued models produced by SSP cbas.

Notice that most of the problems presented in this conference are formalizable in H_{ω_2} or in a fragment of the universe whose theory fall under the scope of the above theorem.
Thus no independence result over MM^{+++} can be proved by means of forcing for all these type of problems.......

What is MM^{+++}

Definition

MM^{+++} states that the class of strongly presaturated towers of normal filters is dense in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$.

Theorem

Assume there are class many strong cardinals δ which are limit of $< \delta$-supercompact cardinals.
TFAE:

1. MM^{+++}
2. $\mathbb{U}_\delta = \mathbb{U}^{\mathsf{SSP},\mathsf{SSP}} \cap V_\delta$ *is a presaturated tower of normal filters for any such* δ.

Fact

If $\mathbb{U}_\delta$ is a a presaturated tower of normal filters, then whenever H is V-generic for $\mathbb{U}_\delta$,

$$\langle H_{\omega_2}^V, \in, P(\omega_1)^V \rangle \prec \langle H_{\omega_2}^{V[H]} \in, P(\omega_1)^V \rangle.$$

Sketch of proof of the generic absoluteness result:

Notice that if $\mathbb{B} \in \mathrm{SSP}$ and G is V-generic for $\mathbb{B}$, in $V[G]$

$$\mathbb{U}_\delta^{V[G]} \text{ is forcing equivalent to } (\mathbb{U}_\delta \restriction \mathbb{B})/G.$$

Assume $\mathbb{B}$ forces MM^{+++}. Pick $\delta > |\mathbb{B}|$ such that $\mathbb{U}_\delta$ is a strong cardinal which is a limit of $< \delta$-spct cardinals.

Let H be V-generic for $\mathbb{U}_\delta$ with $\mathbb{B} \in H$. Then in $V[H]$ there is G V-generic for $\mathbb{B}$ such that $V[H]$ is a generic extension of $V[G]$ for $\mathbb{U}_\delta^{V[G]}$.

Notice that δ is still a strong cardinal which is a limit of $< \delta$-supercompact cardinals in $V[G]$.

Then

$$H_{\omega_2}^V \subset H_{\omega_2}^{V[G]} \subset H_{\omega_2}^{V[H]},$$

$$H_{\omega_2}^V \prec H_{\omega_2}^{V[H]},$$

$$H_{\omega_2}^{V[G]} \prec H_{\omega_2}^{V[H]}.$$

The conclusion follows.

Notice that these results expands on Woodin's generic absoluteness results for $L(\mathbb{R})$ and on (undercover) generic absoluteness results for Σ^1_2-statements which are provable in ZFC.

Lemma

Assume $\phi(r)$ is a Σ^1_2-statement in the real parameter r and $T \supseteq \mathrm{ZFC}$. TFAE

- $T \vdash \phi(r)$.
- *T proves that there is a $\mathbb{B}$ which forces $\phi(r)$.*

Theorem (Woodin)

Assume $\phi(r)$ is a statement in the real parameter r and

$$T \supseteq \mathrm{ZFC} + \text{ there are class many Woodin limit of Woodin.}$$

TFAE:

- $T \vdash \phi(r)^{L(\mathbb{R})}$.
- *T proves that there is a $\mathbb{B}$ which forces $\phi(r)^{L(\mathbb{R})}$.*

Weak versions of these generic absoluteness results and variations of these results which apply also to the class of proper, CCC, axiom A, σ-closed forcings are a current theme of research and work on it has been done by Hamkins, Johnstone, Tsaprounis and others. With Giorgio Audrito we are also looking at these matters in order to get optimal generic absoluteness results also for these classes of forcings and not just for SSP-forcings. With Daisuke Ikegami we are working on the relation between these generic absoluteness results and Woodin's axiom $(*)$.

Thanks for your patience and attention.